Statistical analysis of experimental data
Systemactic uncertainities

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Lecture 09

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Systemactic uncertainities

Iterative procedure (Brandt)

We start from some "initial guess" of parameter values $\mathbf{a_0}$.

Assuming small variations of the model parameters, $\mathbf{a} = \mathbf{a_0} + \delta\mathbf{a}$, we can expand χ^2 in a series:

$$\chi^2(\mathbf{a}) \;=\; \chi^2(\mathbf{a_0}) - \mathbf{b}\cdot(\mathbf{a}-\mathbf{a_0}) + \ldots$$

where $\mathbf{b}$ is the negative gradient of χ^2:

$$\mathbf{b} \;=\; -\nabla\,\chi^2(\mathbf{a_0}) \qquad\qquad b_j \;=\; -\frac{\partial\chi^2}{\partial a_j} \;=\; \sum_{i=1}^{N}\frac{2(y_i-\mu_i)}{\sigma_i^2}\cdot\frac{\partial\mu_i}{\partial a_j}$$

Vector $\mathbf{b}$ defines the direction of steepest χ^2 descent.
One of the possible procedures is to make a step in this direction:

$$\mathbf{a_1} \;=\; \mathbf{a_0} \;+\; \varepsilon\,\mathbf{b}$$

with small $\varepsilon > 0$ and then repeat the whole procedure...

Iterative procedure (Brandt)

We can try to be "smarter". Expanding χ^2 to quadratic term:

$$\chi^2(\mathbf{a}) \quad = \quad \chi^2(\mathbf{a_0}) - \mathbf{b} \cdot (\mathbf{a} - \mathbf{a_0}) + \frac{1}{2}(\mathbf{a} - \mathbf{a_0})^{\mathsf{T}} \mathbb{A} (\mathbf{a} - \mathbf{a_0}) + \ldots$$

where $\mathbb{A}$ is the so called Hessian matrix of second derivatives:

$$\mathbb{A}_{jk} \quad = \quad \left. \frac{\partial^2 \chi^2}{\partial a_j \, \partial a_k} \right|_{\mathbf{a}=\mathbf{a_0}} \approx \sum_{i=1}^{N} \frac{2}{\sigma_i^2} \cdot \frac{\partial \mu_i}{\partial a_j} \cdot \frac{\partial \mu_i}{\partial a_k} \qquad \left(\text{neglecting } \frac{\partial^2 \mu_i}{\partial a_j \, \partial a_k}\right)$$

In this approximation, we can calculate the expected position of the χ^2 minimum:

$$\nabla \chi^2(\mathbf{a}) \quad = \quad - \mathbf{b} + \mathbb{A} \, (\mathbf{a} - \mathbf{a_0}) \quad = \; 0$$

$$\Rightarrow \quad \mathbf{a_m} \; = \; \mathbf{a_0} \; + \; \mathbb{A}^{-1} \mathbf{b}$$

and we can try to "jump" directly to the minimum...

Marquardt Minimization (Brandt)

One of the popular approaches, combining the two previously discussed:

$$\mathbf{a_{i+1}} = \mathbf{a_i} + (\mathbb{A} + \lambda \cdot \mathbb{I})^{-1}\,\mathbf{b}$$

where λ is an additional parameter determining the performance of the algorithm:

- for $\lambda \gg 1$ we make a small step along the gradient direction which corresponds to the gradient minimization with $\varepsilon \approx \frac{1}{\lambda}$
- for $\lambda \ll 1$ we try to “jump” directly to the minimum position Hessian matrix solution is reproduced for $\lambda \to 0$

The key element proposed by D.W.Marquardt (1963) was to use variable λ parameter, adjusting its value to the results of the previous step...

Comparison of variances

If the precision of the measurement is known, we can calculate the χ^2 value resulting from the fit (or arithmetic averaging in the simplest case) to verify the consistency of the procedure (uncertainty estimate in particular).

However, we can also try to compare two independent series of measurements to check, if they are consistent. We can do it even, if our estimate of experimental uncertainties is not very reliable.

One can also consider it as a way to compare two different estimates of the variance of the measurement, and check if they are compatible.

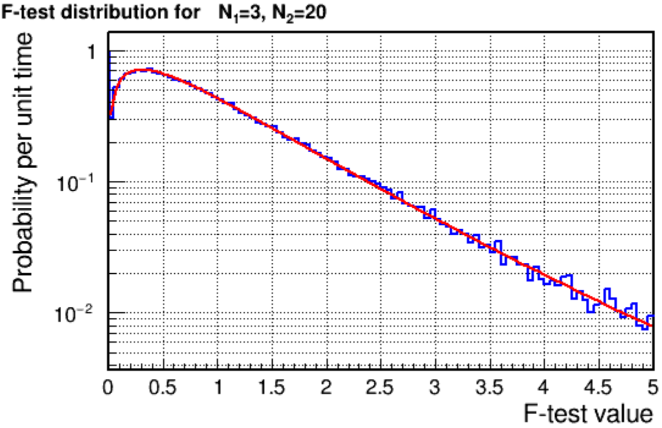
We define the random variable F as: introduced by R.A.Fisher in 1924

$$F = \frac{\chi_1^2/N_1}{\chi_2^2/N_2}$$

where χ_1^2 and χ_2^2 are χ^2 values (or sample variances if we set $\sigma \equiv 1$) of the two independent measurements with N_1 and N_2 degrees of freedom.

F variable distribution

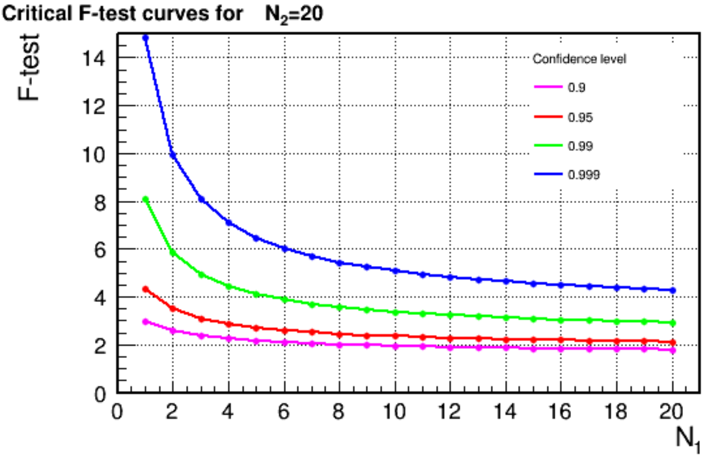
Example distributions of the Fisher's F variable



Significant tail towards high values, even for relatively large N_1, N_2

F-test

Plot of "critical values" of F for N_2 =20



Variations decresase with the size of the reference sample

F-test for fit model (Bonamente)

It turns out that the F variable can also be used to test our fit model.

Let us assume we have a model with m adjustable (free) parameters, which we apply to the set of N data points. Are all model parameters relevant?

We can consider "reduced" version of model with $m - \Delta m$ parameters. It is clear that the resulting χ^2 value will be larger:

$$\chi^2_{(m-\Delta m)} = \chi^2_{(m)} + \Delta\chi^2$$

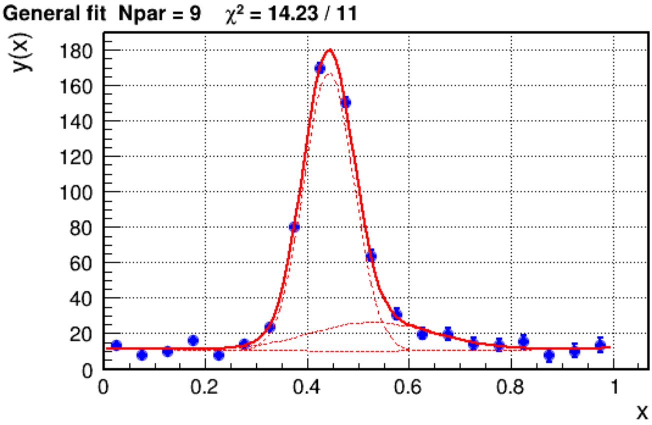
If reduced model is equivalent to the "full" one, distribution of $\Delta\chi^2$ is given by the χ^2 distribution with Δm degrees of freedom. We can test this hypothesis by considering:

$$F = \frac{\Delta\chi^2/\Delta m}{\chi^2_{(m)}/(N-m)}$$

where we need to note that $\chi^2_{(m)}$ and $\Delta\chi^2$ are independent.

F-test for fit model

Example of *F*-test application. We try to fit the data with polynomial background (3 parameters) and two Gaussian peaks (3+3 parameters).



Model constraints

We consider set of N measurement points (x_i, y_i), which can be compared to model predictions depending on parameters $\mathbf{a}$ in terms of the χ^2 value:

$$\chi^2(\mathbf{a}) = \sum_{i=1}^{N} \frac{(y_i - \mu(x_i, \mathbf{a}))^2}{\sigma_i^2}$$

Best estimate of $\mathbf{a}$ should correspond to the minimum of $\chi^2(\mathbf{a})$.

However, we now need to look for this minimum taking set of additional constraints into account:

$$w_k(\mathbf{a}) = 0 \qquad k = 1 \ldots K$$

where number of constraints K should be lower than number of parameters M.

How can we find the best parameter values in this case?

Model reduction

The first approach is to reduce number of model parameters, using constraints to eliminate some of the model variables.
We thus reduce the problem with M model parameters to problem with $M' = M - K$ independent parameters. (method of elements)

Example

We would like to fit polynomial model to a series of measurements where the polar angle $\theta \in [-\pi, +\pi]$ is the controlled variable:

$$\mu(x; \mathbf{a}) = \sum_{k=0}^{M-1} a_k \left(\frac{\theta}{\pi}\right)^k = \sum_k a_k \, x^k$$

where we introduced $x = \frac{\theta}{\pi}$ for simplicity.

And we expect that the distribution should vanish for $\theta \to \pm\pi$:

$$\mu(-1; \mathbf{a}) = \mu(+1; \mathbf{a}) = 0 \qquad K = 2$$

Method of Lagrange Multipliers (Behnke)

The method, invented by J.L.Lagrange in 1788, applies to general minimization problem with additional constraints imposed.

Problem of finding minimum of $\chi^2\ (\mathbf{a})$ with constraints $w_k(\mathbf{a}) = 0$ is equivalent to finding a stationary point (point with all first derivatives at zero) of the Lagrange function:

$$\mathcal{L}(\mathbf{a}, \boldsymbol{\lambda}) \ = \ \chi^2(\mathbf{a}) + \sum_k 2\lambda_k w_k(\mathbf{a})$$

where we introduce additional K parameters λ_k - Lagrange multipliers

Our problem is now reduced to finding parameters **a** and $\boldsymbol{\lambda}$ fulfilling

$$\frac{\partial \mathcal{L}}{\partial a_j} \ = \ 0 \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \lambda_k} \ = \ 0$$

(without any additional constraints)

Method of Lagrange Multipliers

We can write these equations the matrix form:

$$\underbrace{\left(\begin{array}{c|c} \mathbb{A} & \mathbb{D} \\ \hline \mathbb{D}^{\mathsf{T}} & 0 \end{array}\right)}_{\tilde{\mathbb{A}}} \cdot \left(\begin{array}{c} \mathbf{a} \\ \hline \boldsymbol{\lambda} \end{array}\right) = \left(\begin{array}{c} \mathbf{b} \\ \hline \mathbf{c} \end{array}\right)$$

where: $\mathbb{A}_{jk} = \sum_{i=1}^{N} \frac{f_j(x_i)\, f_k(x_i)}{\sigma_i^2}$, $\mathbb{D}_{jk} = d_{k,\,j}$ and $b_j = \sum_{i=1}^{N} \frac{f_j(x_i)\, y_i}{\sigma_i^2}$

and the problem can be solved by inverting matrix $\tilde{\mathbb{A}}$.

Covariance matrix for **a** can be extracted as:

$$(\mathbb{C}_{\mathbf{a}})_{ij} = (\tilde{\mathbb{A}}^{-1})_{ij} \qquad i, j = 1 \ldots M$$

(seems to work for linear problems).

Systemactic uncertainities

Statistical uncertainties

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Uncertainties of the results were related to the fluctuations in the measurement, which can be due to (lecture 01)

- actual nature of the physics process studied
  eg. exponential distribution for decay time measurement
- finite precision of our instruments
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Uncertainties related to fluctuations of the individual measurement results are usually referred to as **statistical uncertainties**.

Systematic uncertainties

Particle physics experiments are quite complex, and so is the data analysis.
We frequently use Monte Carlo methods to correct for different effects.

Simplest example is the (differential) cross section measurement:

$$\boldsymbol{\sigma}_i = \frac{N_i}{\varepsilon_i \, A_i \, \mathcal{L}}$$

where: N_i is the measured number of events (in given bin i), ε - event selection efficiency, A - detector acceptance and $\mathcal{L}$ - integrated luminosity.

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Statistical uncertainty of the extracted cross section value is due to the Poisson fluctuations in the number of reconstructed events.

But we also need to take into account that other factors (ε_i, A_i, $\mathcal{L}$) are also known with finite precision $\Rightarrow$ **systematic uncertainties**

Sources of systematic uncertainties

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Systematic effects are not a problem, if we understand them and know how to model them precisely (correct the final result for systematic error).

Sources of systematic uncertainties

Systematic uncertainty is the uncertainty in the estimation of systematics.

(O.Behnke et. al, *Data Analysis in High Enegy Physics*)
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The maximum-likelihood approach can be used to estimate the impact of systematic effect and the resulting uncertainty of the measurement.
Likelihood function of the procedure used to constrain the systematic effect should be folded into the likelihood function of the main experiment.

Systematic uncertainties

In our simple example of cross section measurement:

$$\boldsymbol{\sigma}_i = \frac{N_i}{\varepsilon_i \, A_i \, \mathcal{L}}$$

the statistical uncertainty on $\boldsymbol{\sigma}_i$ is due to Poisson fluctuations in N_i:

$$\sigma_{stat} = \frac{\sigma_{N_i}}{\varepsilon_i \, A_i \, \mathcal{L}} = \frac{\sqrt{N_i}}{\varepsilon_i \, A_i \, \mathcal{L}} = \frac{\boldsymbol{\sigma}_i}{\sqrt{N_i}}$$

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Uncertainty on $\varepsilon_i \, A_i$ can result from many different sources (including eg. energy calibration), but one should also take into account contribution from the finite statistics of the Monte Carlo events: $r_i = \varepsilon_i \, A_i$

$$\sigma_{r_i} = \sqrt{\frac{r_i(1 - r_i)}{N^{MC}}} \qquad \text{(binomial distribution)}$$

where N^{MC} is the total number of Monte Carlo events (before selection)

Systematic uncertainties

The resulting systematic uncertainty on the cross section measurement:

$$\sigma_{sys(MC)} = \boldsymbol{\sigma}_i \cdot \frac{\sigma_{r_i}}{r_i} = \boldsymbol{\sigma}_i \cdot \sqrt{\frac{1 - r_i}{r_i \, N^{MC}}} = \boldsymbol{\sigma}_i \cdot \sqrt{\frac{1 - r_i}{N_i^{MC}}}$$

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In general, arbitrary level of correlation (when more than one effect is taken into account) is possible for systematic uncertainties...

Systemactic uncertainities

Extended model

We considered outcome of our experiment y_i as a random variable with given probability density function (usually assumed to be Gaussian)

$$f(y_i) = G(y_i; \mu_i, \sigma_i)$$

where, in the general case, the uncertainty of the measurement was given by the (square root of) the variance of the distribution:

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This is how we can define the statistical uncertainty: uncertainty of the measurement when the expected value (and other parameters of pdf) are precisely known:

$$\mu_i = \mu(x_i; \mathbf{a})$$

with controlled variable x_i and all model parameters **a** fixed.

Extended model

To describe systematic effects, we need to introduce additional parameters in the model:

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We usually assume some nominal, expected values of these parameters, $\mathbf{s}_0$.
Uncertainties of these parameters, $\sigma_{\mathbf{s}}$, are then what contributes to the systematic uncertainty of our measurements:

$$\mu_i = \mu(x_i; \mathbf{a}, \mathbf{s}) \;=\; \mu(x_i; \mathbf{a}, \mathbf{s}_0) + \sum_j \frac{\partial \mu_i}{\partial s_j} \cdot (s_j - s_{0,\,j})$$

$$\mu_i \;=\; \mu_0(x_i; \mathbf{a}) + \sum_j \frac{\partial \mu_i}{\partial s_j}\, \sigma_{s_j} \cdot \delta_j \qquad \delta_j = \frac{s_j - s_{0,\,j}}{\sigma_{s_j}}$$

where we introduce variations δ_j scaled to unit normal distribution $(\mu = 0,\ \sigma = 1)$

Extended model

Assuming there is no systematic bias in the measurement (or we already corrected for it), averaging over **s** we should get:

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$$\mathbb{V}(y_i) \; = \; \langle (y_i - \mu_i)^2 \rangle \quad = \quad \langle (y_i - \mu_{0,\,i})^2 \rangle \; + \; \langle (\mu_i - \mu_{0,\,i})^2 \rangle$$

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$$\sigma^2_{y_i} \;=\; \sigma^2_{(stat)\,i} \;+\; \sigma^2_{(sys)\,i}$$

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where we assume independent sources of systematic variations.

Extended model

Covariance matrix for the series of measurements y_i:

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$$= \mathbb{E}((y_i - \mu_{0,\,i})(y_j - \mu_{0,\,j})) + \mathbb{E}((\mu_i - \mu_{0,\,i})(\mu_j - \mu_{0,\,j}))$$

where mixed terms vanish, as systematic variations and statistical fluctuations are independent

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$$= \mathbb{C}_{\mathbf{y}}^{(stat)} + \mathbb{C}_{\mathbf{y}}^{(sys)}$$

where covariance matrix for statistical uncertainties is diagonal:

$$\mathbb{C}_{ij}^{(stat)} = \begin{cases} \sigma^2_{(stat)\,i} & \text{for } i = j \\ 0 & i \neq j \end{cases}$$

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statistical fluctuations are independent

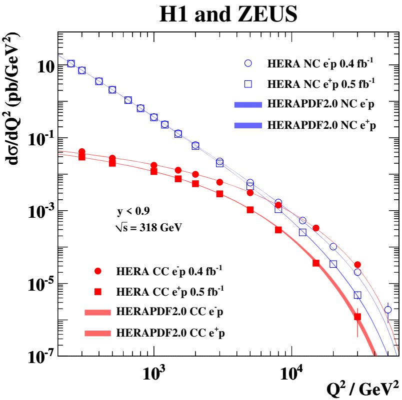
systematic uncertainties result in correlations of expectations:

$$\mathbb{C}_{ij}^{(sys)} = \sum_k \left(\frac{\partial \mu_i}{\partial s_k}\right) \left(\frac{\partial \mu_j}{\partial s_k}\right) \sigma^2_{s_k}$$

We can no longer treat measurements as independent...

Example
SM predictions from HERA

measurement already discussed in lecture 05



NC and CC DIS cross sections comparable for the highest Q^2 values

$$Q^2 \sim M_Z^2, \; M_W^2$$

Combined QCD+EW analysis shows good agreement with SM predictions

Phys. Rev. D 93 (2016) 092002, arXiv:1603.09628

How were systematic uncertainties on the SM predictions calculated?

Example

Let us focus on the "PDF uncertainties", i.e. uncertainties related to our knowledge of the Parton Distribution Functions (PDF) of the proton.

Cross section for NC and CC DIS $e^{\pm}p$ scattering are given in terms of the quark density functions. In the leading order:

$$\frac{d^2\sigma_{CC}^{e^{\pm}p}}{dxdQ^2} = \frac{G_F^2}{4\pi}\left(\frac{M_W^2}{M_W^2+Q^2}\right)^2 \begin{cases} u+c+(1-y)^2(\bar{d}+\bar{s}+\bar{b}) & \text{for } e^-p \\ (1-y)^2(d+s+b)+\bar{u}+\bar{c} & \text{for } e^+p \end{cases}$$

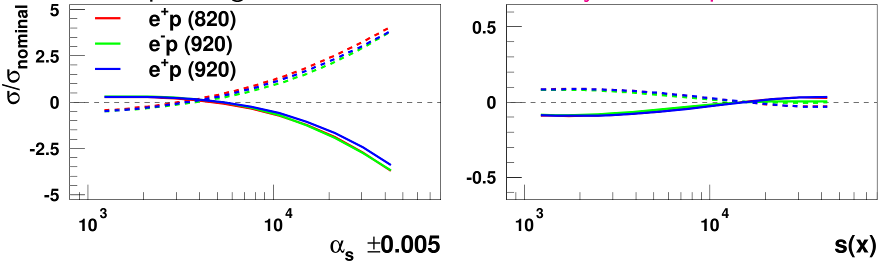
where u, d, s, c, b are quark densities ($\bar{u}, \bar{d} \ldots$ - antiquark) in the proton, extracted by fitting QCD evolution equations to the large set of data from many different experiments (not only DIS).

However, one has to take into account uncertainties of the input data, as well as uncertainties related to different assumptions in the fit...

Example

Considered analysis of HERA data was based on the QCD fit results implemented in EPDFLIB library (M.Botje).

It provided not only the nominal parton density values, but also density values corresponding to variations of different "systematic parameters".



Variations of the three considered NC DIS data sets for up (solid) and down (dashed) variation of different parameters.

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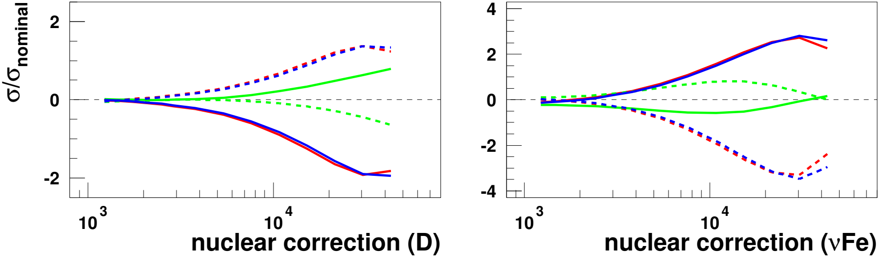

Variations of the three considered NC DIS data sets for up (solid) and down (dashed) variation of different parameters.

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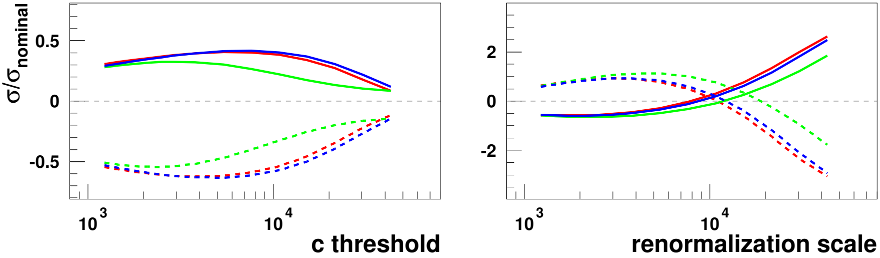

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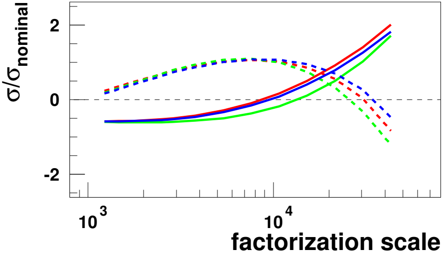
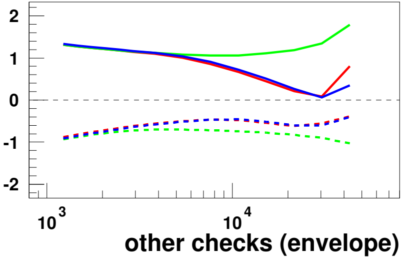
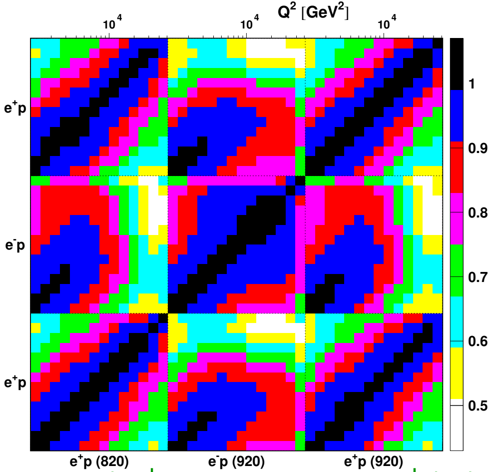




Variations of the three considered NC DIS data sets for up (solid) and down (dashed) variation of different parameters.

Example

Correlation matrix for the expected high Q^2 NC DIS cross sections:



Must be taken into account when we compare our data to SM predictions

Estimating systematic uncertainties

General remarks

One could think that obtaining the proper final result from the analysis (including estimate of the statistical uncertainty of the result) is most important and most difficult. We are almost done...

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Each variable you use in your formula or your analysis code should be considered as a potential source of uncertainty.

But one should also be careful not to overestimate the uncertainties!
Need to distinguish "systematic variations" and "systematic checks"...

Systematic checks

Usually, there are many parameters in the theoretical model or in the detector descriptions which are known with finite precision.

This is the source of systematic uncertainties.

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Final result of our analysis should not depend on these choice, if our approach is valid, but some variations can occur.

One should be very careful! These variations are often due to the finite MC statistics. One should not include them in the systematic uncertainty estimate.
Otherwise, systematic uncertainties can easily "explode" if we use large number of "systematic checks"...

Systemactic uncertainities

Example (Toy model)

An experiment is designed to measure an unknown parameter η.

Two measurements are considered (different experiment configurations) corresponding to two random variables x and y related to the physics parameter η :

$$x_{true} = a + \eta$$
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We can find the optimum way of extracting η by writing down:

$$\chi^2(\eta) = \left(\frac{x - a - \eta}{\sigma_{stat}} \right)^2 + \left(\frac{y - a - 2\eta}{\sigma_{stat}} \right)^2$$

Example

Looking at the minimum of χ^2 we find:

$$0 = \frac{\partial \chi^2}{\partial \eta} = -\frac{2}{\sigma_{stat}^2}\Big(x - a - \eta + 2(y - a - 2\eta)\Big)$$

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This result is not surprising, if we realize that η can be extracted from x and y independently:

$$\eta(x) = x - a \qquad \text{and} \qquad \eta(y) = \frac{1}{2}(y - a)$$

$$\sigma_{\eta(x)} = \sigma_{stat} \qquad\qquad \sigma_{\eta(y)} = \frac{1}{2}\,\sigma_{stat}$$

and minimum of χ^2 corresponds to the weighted average of the two measurements, with uncertainty: $\sigma_y = \frac{1}{\sqrt{5}}\,\sigma_{stat}$

Example

Let us now include systematic variation Δ_{sys} of the background estimate a, so that the expected results of the measurement are

$$\begin{aligned} \langle x \rangle &= x_{true} + \Delta_{sys} \\ \langle y \rangle &= y_{true} + \Delta_{sys} \end{aligned}$$

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$$\eta = \frac{1}{5}\,x + \frac{2}{5}\,y - \frac{3}{5}\,a \quad \Rightarrow \quad \sigma_y^2 = \frac{1}{25}\sigma_{stat}^2 + \frac{4}{25}\sigma_{stat}^2 + \frac{9}{25}\sigma_{sys}^2$$

Including systematic effects

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but is it the optimal procedure?

Example

We should include systematic variation in global likelihood.
Assuming Gaussian uncertainties we get:

$$\chi^2(\eta, \delta) = \left(\frac{x - a - \eta - \delta\sigma_{sys}}{\sigma_{stat}}\right)^2 + \left(\frac{y - a - 2\eta - \delta\sigma_{sys}}{\sigma_{stat}}\right)^2 + \delta^2$$

where δ^2 term corresponds to the likelihood of the systematic variation

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We can calculate partial derivatives to get system of equations:

$$\begin{aligned}
\frac{\partial\chi^2}{\partial\eta} : \quad & 5\eta + 3\sigma_{sys}\delta &= x + 2y - 3a \\
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which we can solve to obtain:

$$\eta = \frac{(2+f^2)y+(1-f^2)x-3a}{5+f^2} \quad \text{where} \quad f = \frac{\sigma_{sys}}{\sigma_{stat}}$$

Example

For small systematic uncertainties $(\sigma_{sys} \ll \sigma_{stat},\ f \ll 1)$

$$\eta \;=\; \frac{(2+f^2)y + (1-f^2)x - 3a}{5+f^2} \;\rightarrow\; \frac{2y + x - 3a}{5}$$

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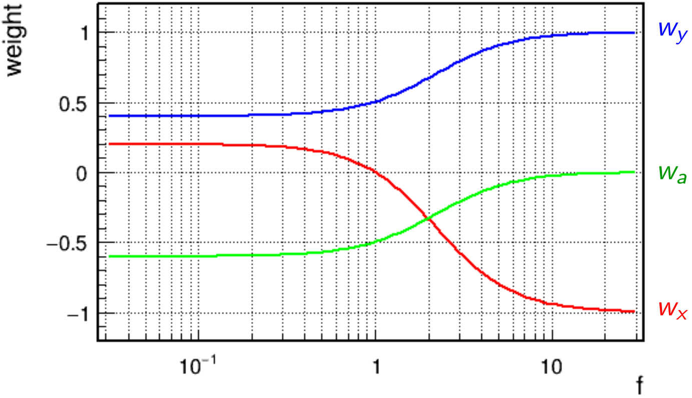
It is also interesting to note that for $(\sigma_{sys} = \sigma_{stat},\ f = 1)$, measurement of x is not used:

$$\eta \;=\; \frac{(2+f^2)y + (1-f^2)x - 3a}{5+f^2} \;=\; \frac{y - a}{2}$$

Example

Weights of the two measurements and background estimate



$$\eta = w_x\, x + w_y\, y + w_a\, a$$

Example

How about uncertainty of the extracted η value?

We can obtain it from the covariance matrix:

$$\mathbb{C}_{(\eta,\delta)} = \left(\frac{1}{2} \frac{\partial^2 \chi^2}{\partial a_l \, \partial a_k} \right)^{-1} = \begin{pmatrix} \frac{5}{\sigma^2_{stat}} & \frac{3\sigma_{sys}}{\sigma^2_{stat}} \\ \frac{3\sigma_{sys}}{\sigma^2_{stat}} & \frac{2\sigma^2_{sys}}{\sigma^2_{stat}} + 1 \end{pmatrix}^{-1}$$

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for $f \to \infty$ $\quad \to \sqrt{2}\,\sigma_{stat} \quad\quad \to \infty$

General procedure

General procedure for including systematic uncertainties in the analysis is to consider corresponding systematic shifts as additional model parameters

$$\mu_i = \mu(x_i; \mathbf{a}, \mathbf{s})$$

$$\chi^2(\mathbf{a}, \mathbf{s}) = \sum_{i=1}^{N} \frac{(y_i - \mu(x_i, \mathbf{a}, \mathbf{s}))^2}{\sigma_i^2} + \sum_{k=1}^{K} \frac{(s_k - s_{0,\,k})^2}{\sigma_{s_k}^2}$$

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If systematic parameters are not independent (are correlated)

$$\chi^2(\mathbf{a}') = \sum_{i=1}^{N} \frac{(y_i - \mu(x_i, \mathbf{a}'))^2}{\sigma_i^2} + \sum_{k,j} (s_k - s_{0,\,k})(s_j - s_{0,\,j}) \left(\mathbb{C}_{\mathbf{s}}\right)^{-1}_{j,k}$$

General procedure

χ^2 minimization procedure is basically unchanged, only the additional terms (systematic constrains) need to be included in calculations (as for the parameter constraints).

Negative gradient of χ^2 uncorrelated systematics

$$b_j = -\frac{\partial \chi^2}{\partial a'_j} = \sum_{i=1}^{N} \frac{2(y_i - \mu_i)}{\sigma_i^2} \cdot \frac{\partial \mu_i}{\partial a'_j} - \frac{2(s_j - s_{0,j})}{\sigma_{s_j}^2}$$

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where systematic shifts **s** are assumed to go first in **a**' (for proper indexing)

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Negative gradient of χ^2 general case

$$b_j = -\frac{\partial \chi^2}{\partial a'_j} = \sum_{i=1}^{N} \frac{2(y_i - \mu_i)}{\sigma_i^2} \cdot \frac{\partial \mu_i}{\partial a'_j} - 2 \sum_k (s_k - s_{0,\,k}) \, (\mathbb{C}_{\mathbf{s}})^{-1}_{j,k}$$

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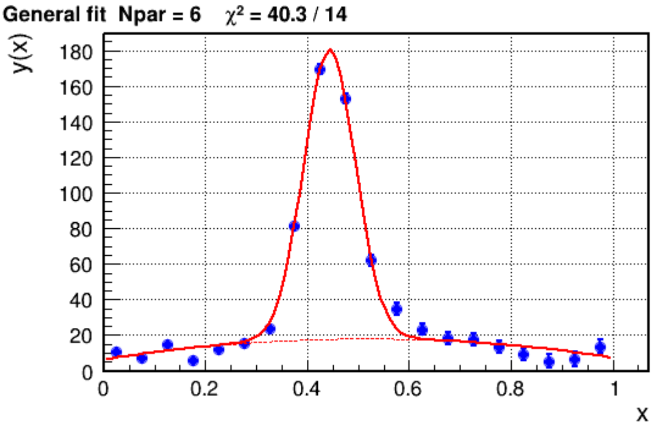
General procedure example

Fitting Gaussian peak on top of background (lecture 08)



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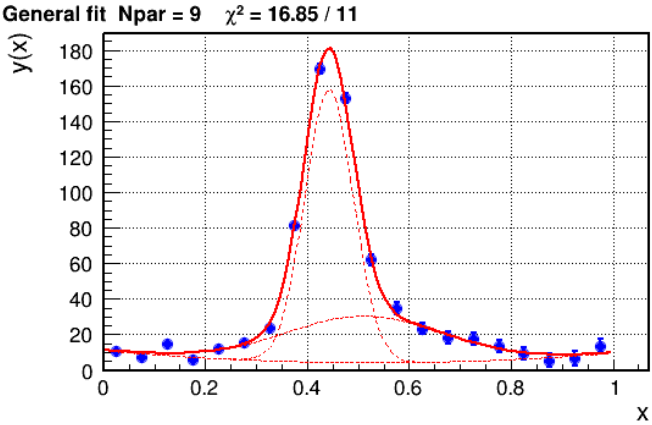

Two peak fit is better, but improvement not very significant, $p = 0.02$

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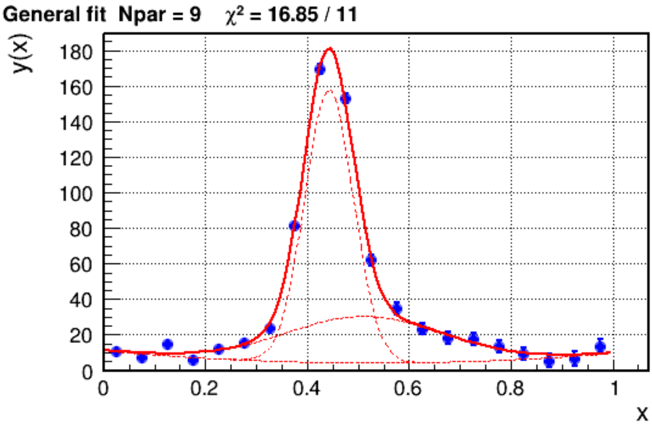
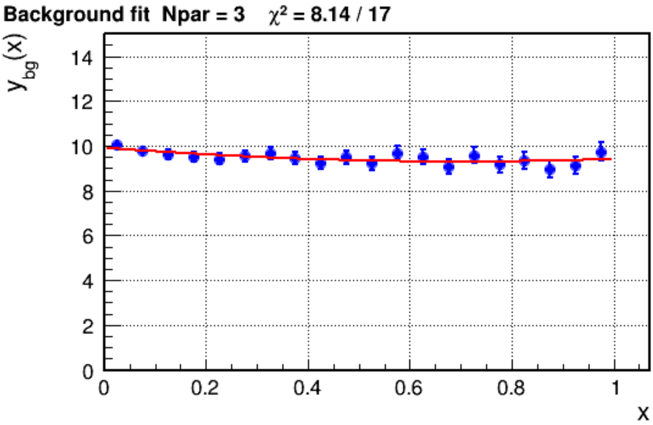


But we also see that background fit changes a lot...

General procedure example

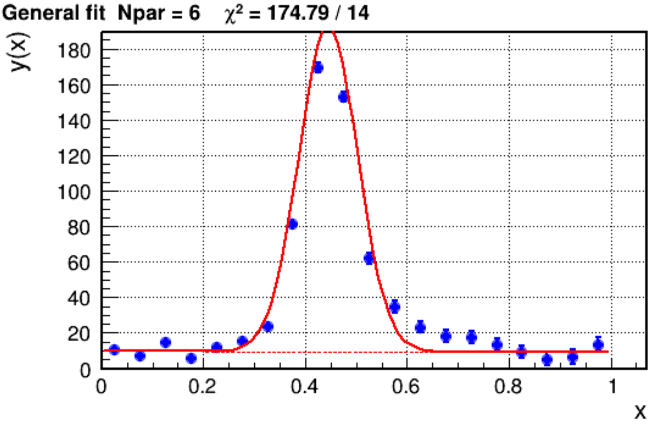
Suppose we can perform an independent background measurement with higher precision and fit parameters of our background model



General procedure example

We can now use parameters from the background fit in signal fit

same data sample as before



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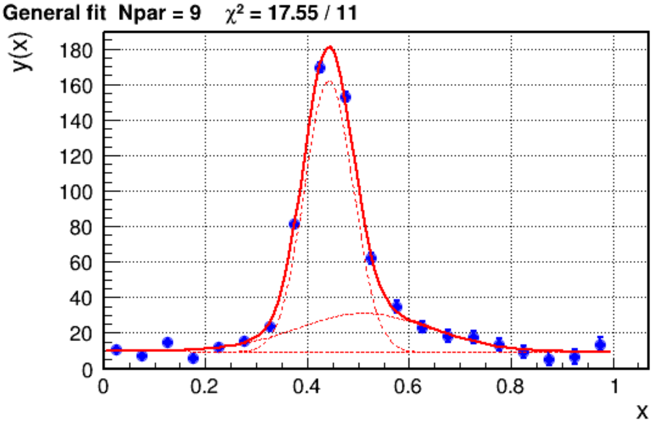

Second peak significance increase from 2.1σ to 4.3σ ($p = 0.9 \cdot 10^{-5}$)

Systemactic uncertainities

Reducing variables

Problem

In the general case, one can consider a huge number of systematic effects, each will contribute to the final systematic uncertainty.

Number of systematic effects can be larger than the number of relevant model parameters (which we want to extract) or even the number of measurements.

Systematic uncertainties of our measurements are (in most cases) correlated, so one needs to use the full covariance matrix.

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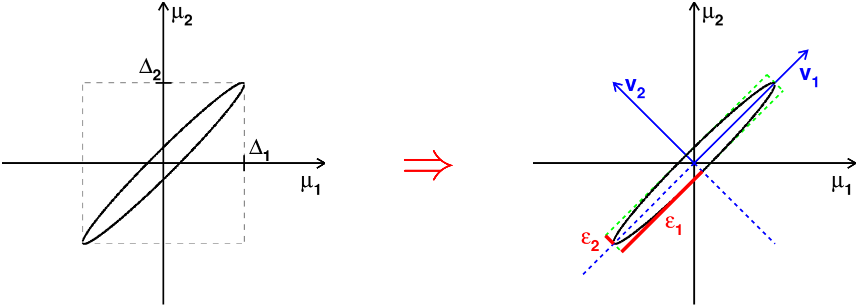
Is there a way to simplify the problem?

Is there a way to reduce the number of systematic variations to consider?
This is also important when we want to model the experiment (eg. with Monte Carlo methods)

Eigenvectors

Correlations between variables can be removed througth 'rotation' in the variable space.



This is a problem of finding "eigenvectors" of the covariance matrix.
Directions such that:

$$\mathbb{C}_{\mathbf{s}} \cdot \mathbf{v} \quad = \quad \sigma_{\mathbf{v}}^2 \cdot \mathbf{v}$$

Reducing variables

Eigenvectors

Eigenvectors of the covariance matrix define "uncorrelated directions" in the space of systematic parameter variations.

Variations along these directions are independent (uncorrelated).
We can redefine our systematic variables to remove correlations...

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Eigenvalues

Eigenvalues give us the size of variations along given eigenvector ($\sigma_{\mathbf{v}}^2$)
$\Rightarrow$ we can tell what variations are most relevant

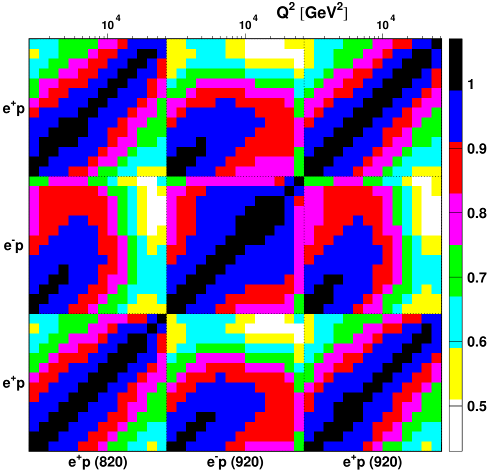
By identifying variations which give leading contributions to the covariance matrix, we can limit number of variations considered in our problem.

Variations corresponding to eigenvectors with very small eigenvalues can be safely ignored...

We assume eigenvectors are ordered from highest to lowest eigenvalue.

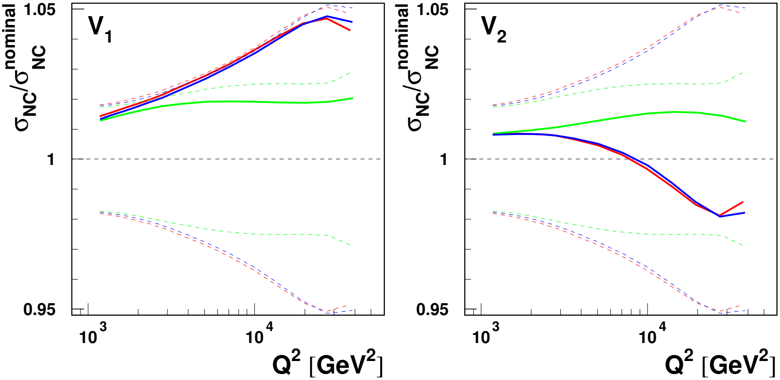
Eigenvectors

Let us consider uncertainties of the high Q^2 NC DIS cross sections again.
Correlation matrix:



Eigenvectors

Systematic variations corresponding to eigenvectors of correlation matrix:

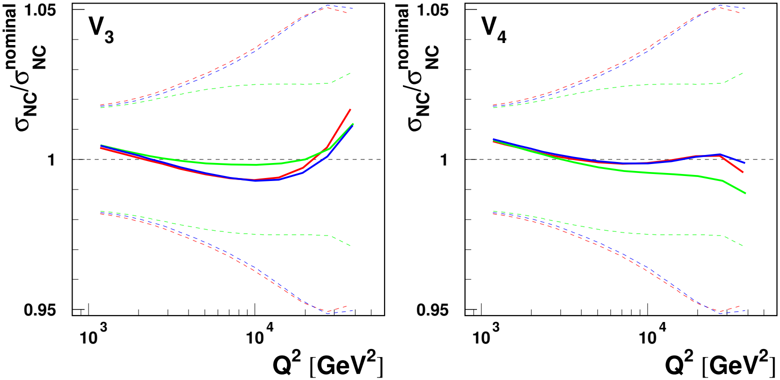


Variations relative to the nominal SM expectations

Dominant contribution from the first eigenvector...

Eigenvectors

Systematic variations corresponding to eigenvectors of correlation matrix:

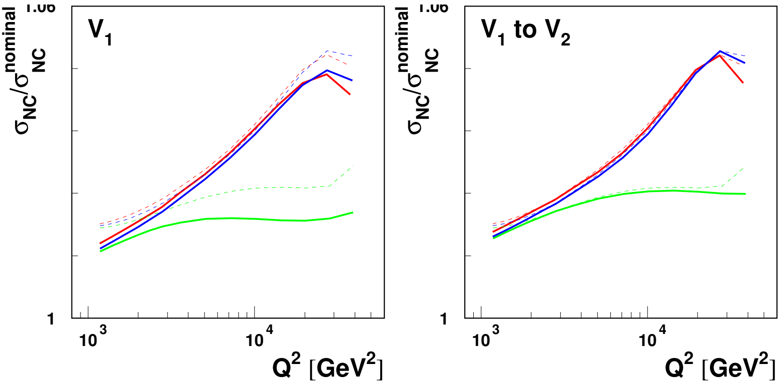


Variations relative to the nominal SM expectations

Dominant contribution from the first eigenvector...

Eigenvectors

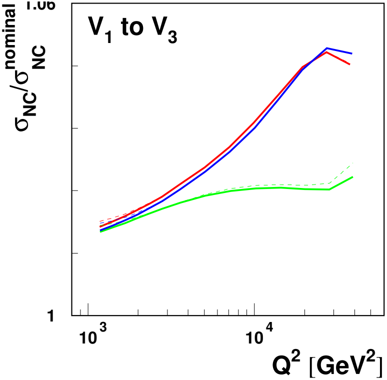
Variations corresponding to the sum of eigenvector contributions:



Variations relative to the nominal SM expectations

Eigenvectors

Variations corresponding to the sum of eigenvector contributions:



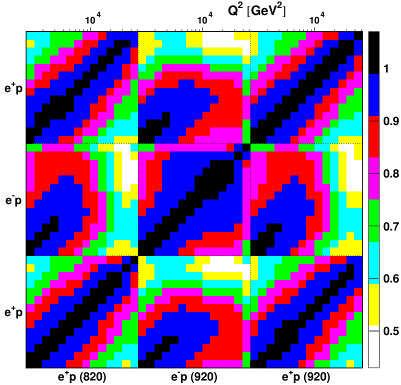
Variations relative to the nominal SM expectations

Four first eigenvectors perfectly reproduce total systematic uncertainty

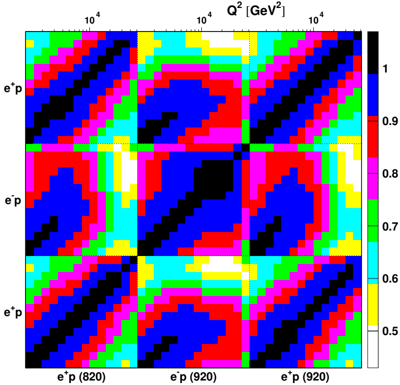
Eigenvectors

Correlation matrix comparison:

Full matrix



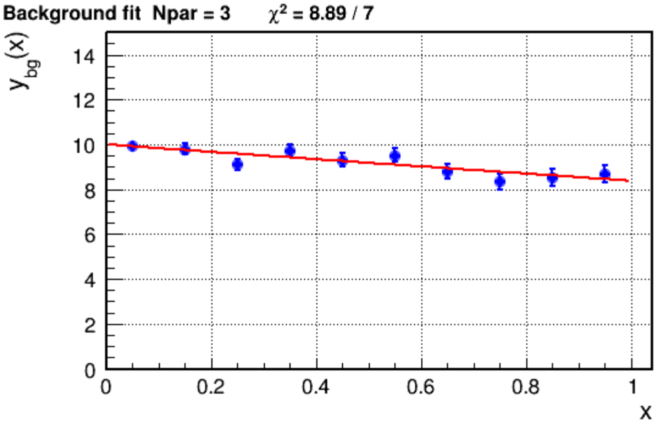
Four eigenvectors



Correlations between variables also very well reproduced

Eigenvectors

Example problem: eigenvectors of background covariance matrix



Background expectations from the fit are correlated between points...

Systemactic uncertainities

Homework Solutions to be uploaded by January 12.

CLIC accelerator should allow to use polarized electron beam, $P_e = \pm 0.8$

Consider measurement of signal events in the presence of background, which is polarization independent:

$$N_{exp} \;=\; \mathcal{L}\left(\boldsymbol{\sigma}_{bg} + (1 - P_e)\cdot\boldsymbol{\sigma}_{sig}\right)$$

(we assume $A\,\varepsilon = 1$ for simplicity).

Assume that polarization P_e is known with very high precision, the number of events collected is large ($N_{exp} \gg 1$) and $\boldsymbol{\sigma}_{sig}$ of the same order that $\boldsymbol{\sigma}_{bg}$

Calculate the optimal sharing of the total luminosity between runs with negative and positive polarization, as a function of the theoretical uncertainty on the background level, $\Delta = \frac{\sigma_{\boldsymbol{\sigma}_{bg}}}{\boldsymbol{\sigma}_{bg}}$ (assumed to be Gaussian). Discuss the asymptotic cases.

Optimal: resulting in most precise estimate of $\boldsymbol{\sigma}_{sig}$