Statistical analysis of experimental data
Probability distributions and their properties

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Lecture 03

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Probability distributions and their properties

Definition of Probability

Classical definition as developed in the 18th–19th centuries

If the sample space contains N_Ω elementary events (possible outcomes of the experiment) and the considered event A contains N_A elementary events, then, **assuming all elementary events are equally probable**

$$P(A) = \frac{N_A}{N_\Omega}$$

Only applicable to "uniform" sample spaces (eg. gambling games).

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Frequentist definition

When repeating the same experiment a large number of times, $N \gg 1$, the probability of A

$$P(A) = \lim_{N\to\infty} \frac{N(A)}{N}$$

where $N(A)$ is the number of occurrences of the event A

Probability does depends on the definition of the considered sample space!

Probability distributions and their properties

Experiments

So far, we have considered probability as a very general concept.
We only assumed that an experiment delivers data which are a subject to fluctuations.
But we did not look at the details of the obtained data.

The outcome of the experiment can be of different nature:

- observation (or non-observation) of given process (true/false)
- observation of an event from given category (classification)
- number of the occurrences of given event (counting)
- value of given observable (measurement)

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- number of the occurrences of given event (counting)
- value of given observable (measurement)

We usually present the outcome of the experiment (measurement) in a numerical form.
Two general types of variables can be considered:

- discrete (logical, classification and counting)
- continuous (measurement)

Experiments

When repeating the experiment many times, numerical results fluctuate, reflecting fluctuations of the measurement (see lecture 01).

The numerical result of a repeated experiment (measurement) can not be predicted,
it is only known when the experiment is made

$\Rightarrow$ that is why we call it a random variable

The true value of the considered physical parameter is usually unknown
or known with limited precision only.

By repeating the measurement many times we typically want to increases our knowledge of this parameter, estimate its value with the highest possible precision.

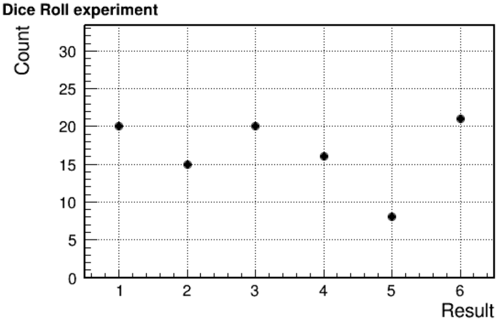
We can also repeat the measurement to better understand the measurement process itself and find the proper description of the observed fluctuations.

Distributions

It is very practical to present results of a repeated experiment in a form of a distribution of the considered random variable.

For a discrete variable: plot the count number for each elementary event.

Example of dice roll experiment: 100 rolls, 1^{st} try



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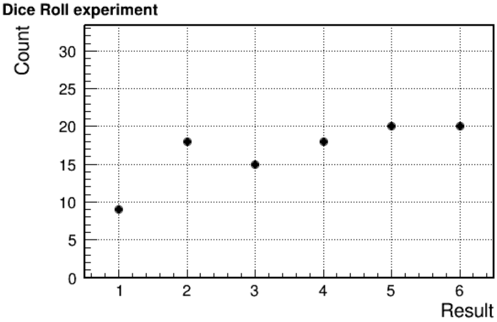

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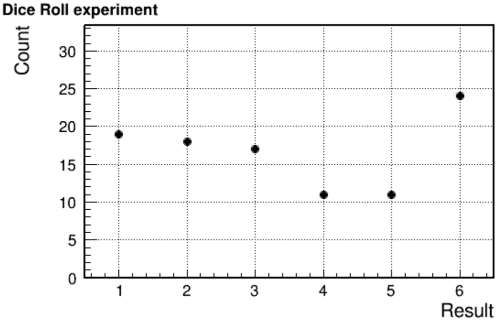
Example of dice roll experiment: 100 rolls, 3^{rd} try



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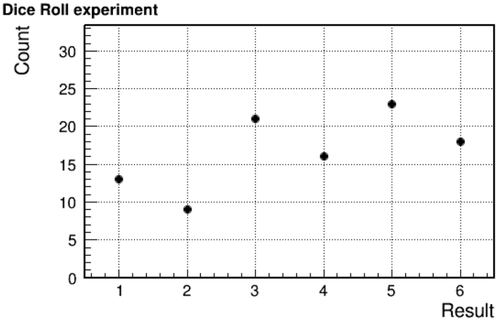
Example of dice roll experiment: 100 rolls, 4^{th} try



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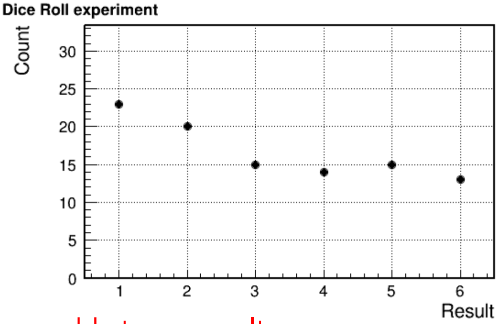
Example of dice roll experiment: 100 rolls, 5^{th} try



Significant fluctuations observed between results

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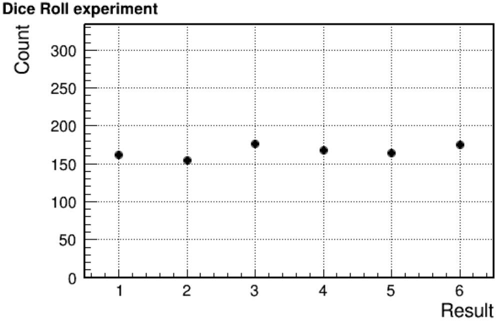
Example of dice roll experiment: 1000 rolls



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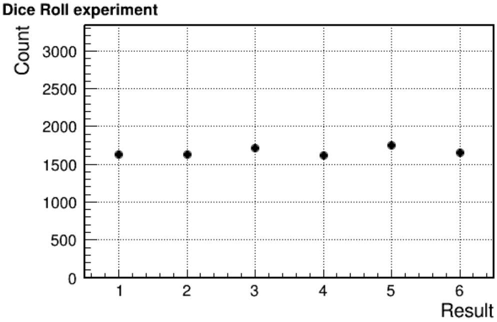
Example of dice roll experiment: 10000 rolls



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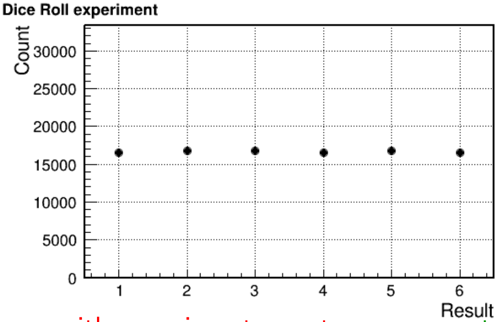

Relative fluctuations decrease with experiment count as expected in frequentist appoach

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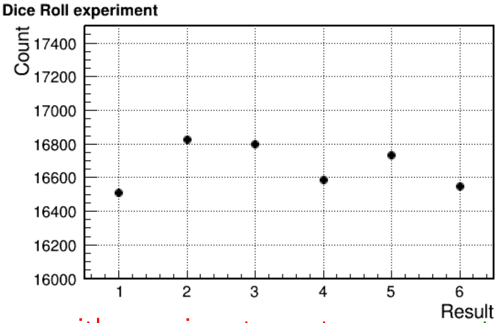
Example of dice roll experiment: 100000 rolls (zoom)



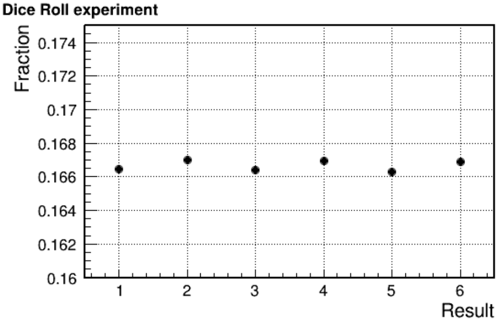
Relative fluctuations decrease with experiment count as expected in frequentist appoach

Distributions

It is very practical to present results of a repeated experiment in a form of a distribution of the considered random variable.

We can plot also plot the result as the relative fraction.

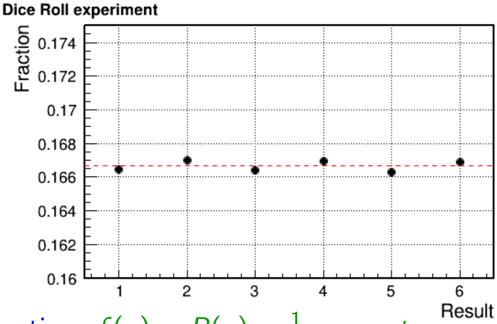
Example of dice roll experiment: 1000000 rolls



Probability distribution function

In the limit of the infinite number of experiments, the relative fraction is given by a probability distribution function (PDF) (frequentist definition)

For discrete variables, probability distribution function is the probability that a given value of the random variable occurs in a single experiment.



Probability distribution function: $f(n) = P(n) = \frac{1}{6} = const$

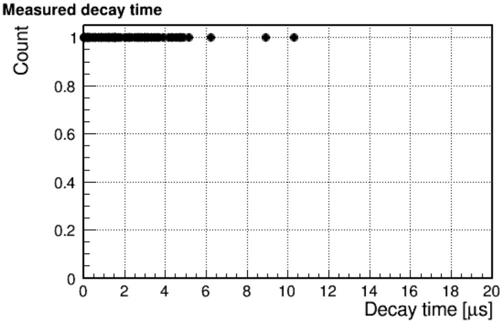
Histograms

Graphical presentation becomes more difficult for continuous variable.

With high readout precision, probability of obtaining the same numerical result twice is negligible.

The method of plotting the count number for each result does not work!

Example of decay time measurement: $\tau = 2.2\mu s$, 100 decays



Histograms

Instead of asking for given values, we need to look at defined value ranges.

We usually define a set of value bins covering the whole considered value range and count events when variable value is in given bin.

Then we can plot the count number for each bin

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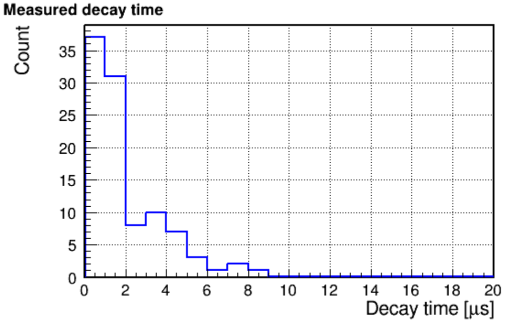

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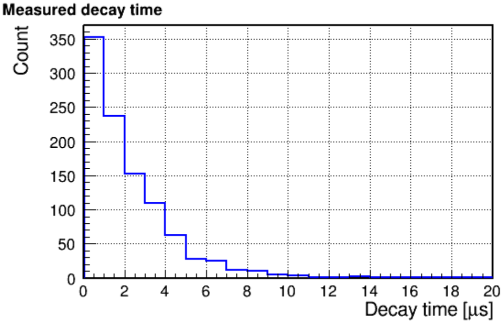

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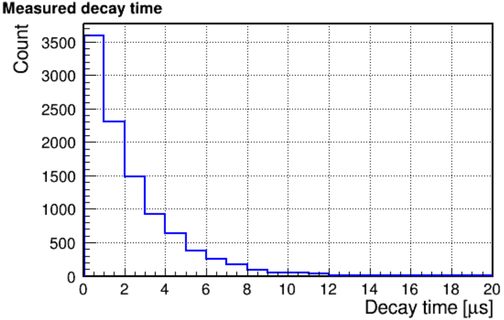

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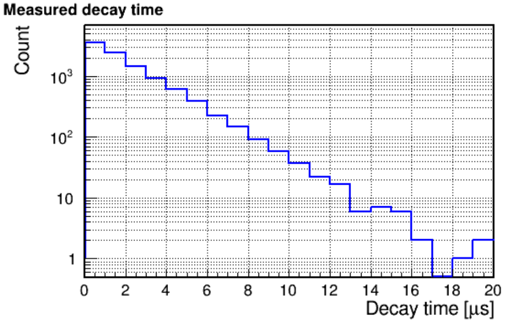

Probability distribution function

We can calculate the relative fraction of events in each bin.

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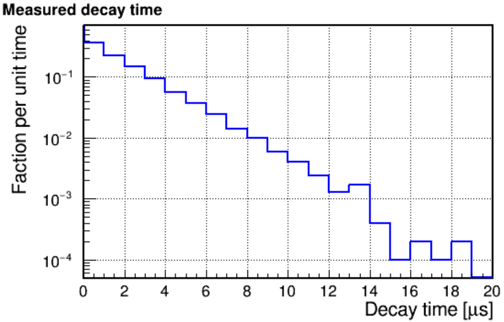

Probability distribution function

We can calculate the relative fraction of events in each bin.

In the limit of the infinite number of experiments (and very narrow bins), the relative fraction is given by a probability distribution function (PDF) of the variable multiplied by the bin width.

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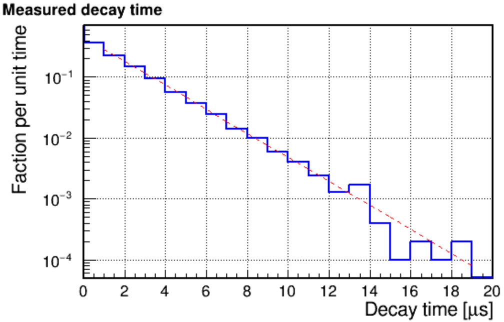

Probability distributions and their properties

Probability distributions

Probability distribution function (PDF)
also called "Probability density function" in some books

For given random variable X, probability distribution function, $f(x)$, describes the probability to obtain given numerical result x (in single experiment). For infinitesimal interval dx:

$$P(x < X < x + dx) \quad = \quad f(x)\,dx$$

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For arbitrary interval $[x_1, x_2]$:

$$P(x_1 < X < x_2) \quad = \quad \int_{x_1}^{x_2} dx\, f(x)$$

This can be considered an alternative definition of $f(x)$

Cumulative distribution function

We can also define cumulative distribution function $F(x)$, which is the probability that an experiment will result in a value not grater than x:

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Probability that the X value observed is in the range from x_1 to x_2:

$$P(x_1 < X < x_2) \quad = \quad F(x_2) - F(x_1) \; = \; \int_{x_1}^{x_2} dx\, f(x)$$

General properties of distribution functions

From the properties of probability

$$f(x) \geq 0 \qquad\qquad \int_{-\infty}^{+\infty} dx\, f(x) = 1$$

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For cumulative distribution:

$$F(x) \;=\; \int_{-\infty}^{x} dx'\, f(x')$$

$$\Rightarrow \quad \lim_{x \to -\infty} F(x) = 0 \qquad\qquad \lim_{x \to +\infty} F(x) = 1$$

Moments of distribution functions

Expectation value of an arbitrary function $g(x)$ of the random variable X can be defined as

$$\mathbb{E}(g(x)) = \langle\, g(x) \,\rangle = \int_{-\infty}^{+\infty} dx\; g(x)\; f(x)$$

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The expectation value of a random variable itself or **the mean**:

$$\mu = \mathbb{E}(X) = \langle\, x \,\rangle = \bar{x} = \int_{-\infty}^{+\infty} dx\; x\; f(x)$$

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For discrete random variables mean is given by the sum of all possible values x_i of X multiplied by their corresponding probabilities.

Moment of order n (n^{th} moment) is defined as

$$\mu_n = \mathbb{E}(X^n) = \langle x^n \rangle = \int_{-\infty}^{+\infty} dx \, x^n \, f(x) \qquad = \sum_i x_i^n \, f(x_i)$$

Mean value is, by definition, the first ($n = 1$) moment of the probability distribution, $\mu \equiv \mu_1$.

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Central moment of order n is defined as

$$m_n = \mathbb{E}\left((X - \mu)^n\right) = \langle (x - \mu)^n \rangle = \int_{-\infty}^{+\infty} dx\, (x - \mu)^n\, f(x)$$

$$= \sum_i (x_i - \mu)^n\, f(x_i)$$

By calculating moments of the (unknown or known with limited precision) probability distribution we can extract information about its shape.

Moments of distribution functions

For the lowest order moments we have:

$$\mu_0 \equiv 1 \qquad m_0 \equiv 1 \qquad \text{normalization of } f(x)$$
$$\mu_1 \equiv \mu \qquad m_1 \equiv 0 \qquad \text{definition of mean value of } f(x)$$

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The first moment which gives us information about the shape of $f(x)$ is **Variance**, which is the second central moment:

$$\mathbb{V}(X) = m_2 = \langle (x-\mu)^2 \rangle = \int_{-\infty}^{+\infty} dx \, (x-\mu)^2 \, f(x)$$

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$$= \sum_i (x_i - \mu)^2\, f(x_i)$$

The square root of the variance is referred to as the standard deviation σ.

Describes the average difference between measurements x_i and their mean μ.

Probability distributions and their properties

Resources

Possible places to look for numerical tools:

- basic distributions are just available in NumPy:
  https://numpy.org/doc/stable/reference/random/generator.html#distributions

- for more complete list of possible probability distributions you can look at SciPy:
  https://docs.scipy.org/doc/scipy/reference/stats.html#probability-distributions

Consider an experiments with only two possible outcomes
(binary experiment): $\Omega = \{\text{'success'}, \text{'failure'}\}$ (or $\Omega = \{\text{'true'}, \text{'false'}\}$)

This is also the case, when we look for particular event A in wider sampling space.

Binomial distribution

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What is the probability of having n successes in N tries?
We are not interested in the order in which the successes take place.

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Example

What is the probability to get three 'six' when rolling the dice five times?

$$p = \frac{1}{6} \qquad n = 3 \qquad N = 5$$

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It is important to notice that we ask for a probability for an event from a different, extended sampling space, $\Omega' = \Omega^N$!

Binomial distribution

Describes probability of having n successes in N tries, assuming success probability p in single trial and failure probability $q = 1 - p$

$$P(n) \;=\; \binom{N}{n}\, p^n \, q^{N-n} \;=\; \frac{N!}{n!(N-n)!}\, p^n \, q^{N-n}$$

The Newton symbol $\binom{N}{n}$ gives the number of possible sequences of N tries giving n successes (regardless of order) and $p^n \, q^{N-n}$ describes the probability for single such sequence (elementary event).

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Mean (expected value) of the binomial distribution

$$\langle n \rangle = \bar{n} = p \, N$$

Variance of the distribution

important for efficiency uncertainty

$$\sigma^2 = p \, (1 - p) \, N$$

For given N, distribution is widest for $p = 0.5$

Example (1)

03_binomial.ipynb Open in Colab

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$$P(n) = \frac{N!}{n!(N-n)!}\, p^n\, q^{N-n} = \frac{5!}{3!\,2!}\,\frac{1}{6^3}\,\frac{5^2}{6^2} = \frac{120 \cdot 1 \cdot 25}{6 \cdot 2 \cdot 216 \cdot 36} \approx 0.032$$

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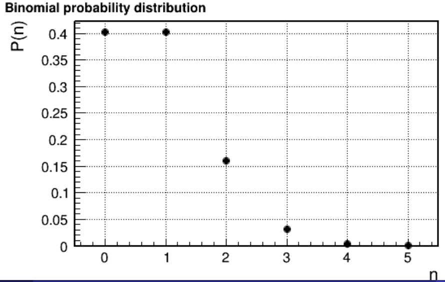

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Probability that students will NOT fit in the room:

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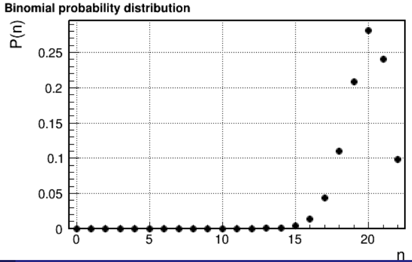

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There is 66% chance that the room will be large enough for all students.

But this probability applies to single lecture only!
Probability that they will fit in the room for 14 lectures is

$$P^{OK} = (1 - P^{ovfl})^{14} \approx 0.003$$

$\Rightarrow$ we can hardly count on luck in this case, we need larger room !

Example (3)

Consider results of the quality test at the factory.
Out of N tested products of a given kind, N_1 passed the quality check and N_0 failed.

Efficiency of the production process (probability of positive test) can be estimated as

$$\varepsilon = \frac{N_1}{N}$$

Example (3)

Consider results of the quality test at the factory.

Out of N tested products of a given kind, N_1 passed the quality check and N_0 failed.

Efficiency of the production process (probability of positive test) can be estimated as

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This formula remains valid also for variable N...

Example (3)

Mistake quite often made by student is to calculate the uncertainty of $\varepsilon = \frac{N_1}{N}$ assuming N_1 and N are two independent variables with Poisson uncertainty estimated as:

$$\sigma^2_{N_1} = N_1 \qquad \sigma^2_N = N$$

which results in the final uncertainty estimate: we will discuss error propagation later

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$$\sigma^2_\varepsilon = \frac{\varepsilon \cdot (1+\varepsilon)}{N}$$ **WRONG**

One can immediately see that there is a problem considering the case of $N_1 \sim N$.
For $N = 1000$ and $N_1 = 999$ we get $\varepsilon = 0.999$ and $\sigma_\varepsilon \approx 0.045$, which suggests much higher level of fluctuations than seems possible...

This result is wrong because N and N_1 are not independent! N_1 is always a subset of N

Example (3)

Correct approach to the problem (when we do not want to use properties of the binomial distribution) is to rewrite the efficiency formula as

$$\varepsilon = \frac{N_1}{N_1 + N_2}$$

and assume that N_1 and N_2 are two independent variables with Poisson uncertainty. The final uncertainty estimate resulting from error propagation is then given by:

$$\sigma_\varepsilon^2 = \frac{N_1 \cdot N_2}{(N_1 + N_2)^3} = \frac{\varepsilon \cdot (1 - \varepsilon)}{N}$$ same as from binomial distribution

which has very different behaviour for $N_1 \to N$. For $N = 1000$ and $N_1 = 999$ we get $\varepsilon = 0.999$ and $\sigma_\varepsilon \approx 0.001$, which is what we would expect (from expected fluctuations in N_2)

Uniform probability distribution

Is often used as a model for a "complete randomness" of measurement result in given range. If variable x is restricted to interval $[a, b]$:

$$f(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{for } x > b \end{cases}$$

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Mean (expected value) of the uniform distribution

$$\bar{x} = \langle x \rangle = \frac{a+b}{2}$$

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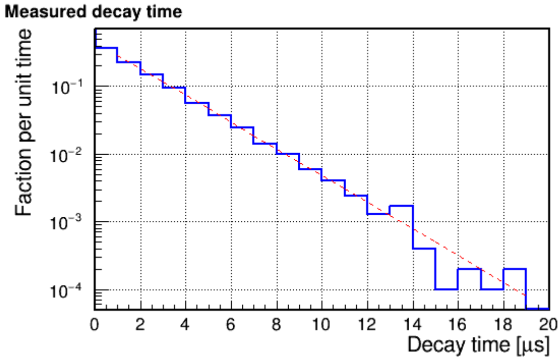
Variance of the uniform distribution

$$\mathbb{V}(x) = \sigma^2 = \frac{(b-a)^2}{12}$$

Exponential probability distribution

Describes the probability of waiting time t, when we wait for event A and the probability of A in a small time interval dt is constant: $dp = dt/\tau$.

This is the case for particle and nuclear decays, but also for other phenomena τ is the only parameter



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$$F(t+dt) = F(t) + (1 - F(t)) \cdot \frac{dt}{\tau}$$

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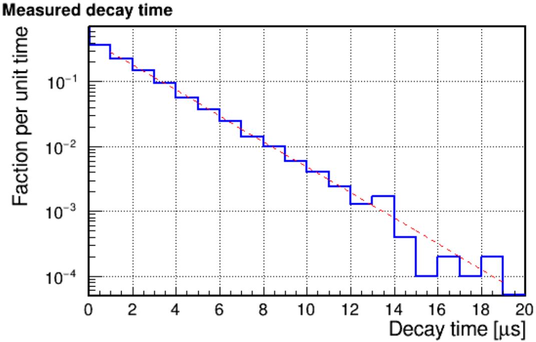
\frac{d}{dt} (1 - F(t)) &= -\frac{1}{\tau} \cdot (1 - F(t)) \\
(1 - F(t)) &= C \cdot e^{-t/\tau} \\
F(t) &= 1 - e^{-t/\tau} \qquad C = 1 \ \text{ from boundary conditions}
\end{aligned}
$$

Exponential probability distribution

Resulting formula for the probability distribution is:

$$f(t) = \begin{cases} \frac{1}{\tau} \cdot e^{-t/\tau} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

indicated by the red dashed line:



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$$\langle\, t \,\rangle \;=\; \int_0^{+\infty} dt\; t\; f(t) \;=\; \tau \qquad \text{integrating by parts}$$

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Variance of the exponential distribution

$$\mathbb{V}(t) \;=\; \sigma^2 \;=\; \int_0^{+\infty} dt\; (t-\tau)^2\; f(t) \;=\; \tau^2$$

Example

What is the probability that the particle does not decay within 10 lifetimes?

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We can just look at the cumulative distribution:

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\begin{aligned}
F(t) &= 1 - e^{-t/\tau} \\
1 - F(t) &= e^{-t/\tau} \\
1 - F(10\tau) &= e^{-10} \approx 0.0000454
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Half-life

Frequently used in nuclear physics, nuclear medicine etc.
Defined as a time needed for half of the nuclei to decay.

$$
\begin{aligned}
F(t_{1/2}) &= 0.5 \\
\Rightarrow \quad t_{1/2} &= \ln 2 \cdot \tau
\end{aligned}
$$

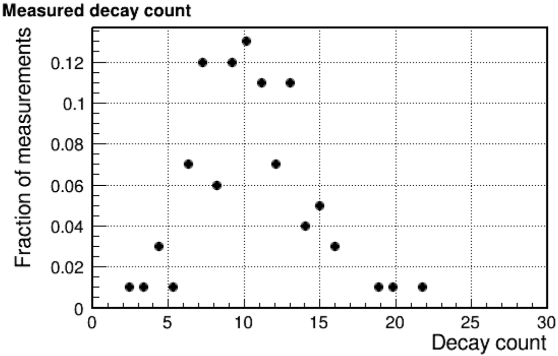
Expected number of decays

03_counting.ipynb Open in Colab

Consider radioactive source with 1 decay per second (1 Bq).

What is the expected result of the experiment counting decays in 10 s time window?

Example of decay count measurement: 100 measurements (100 × 10 s)



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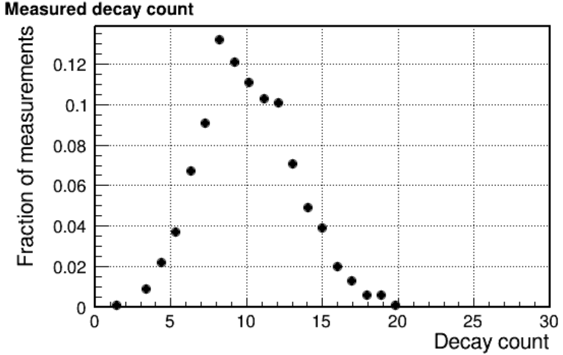
Example of decay count measurement: 1000 measurements



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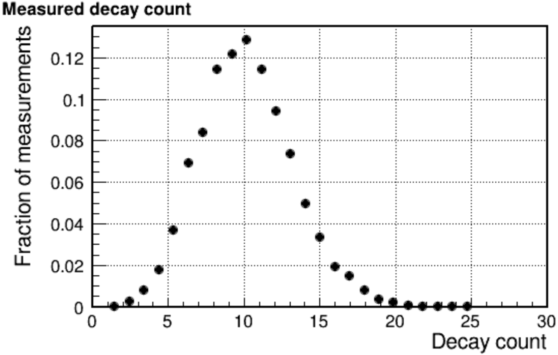
Example of decay count measurement: 10000 measurements



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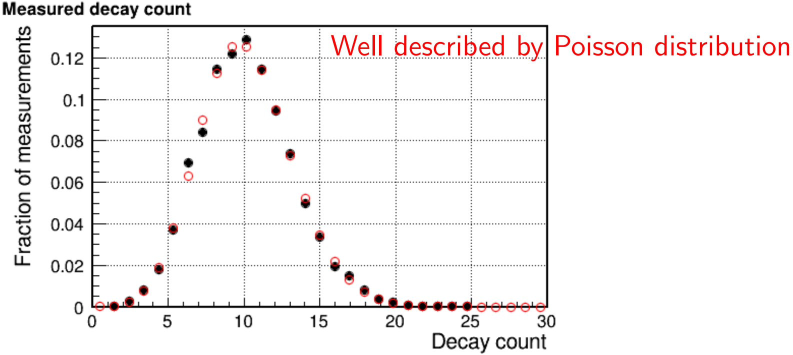

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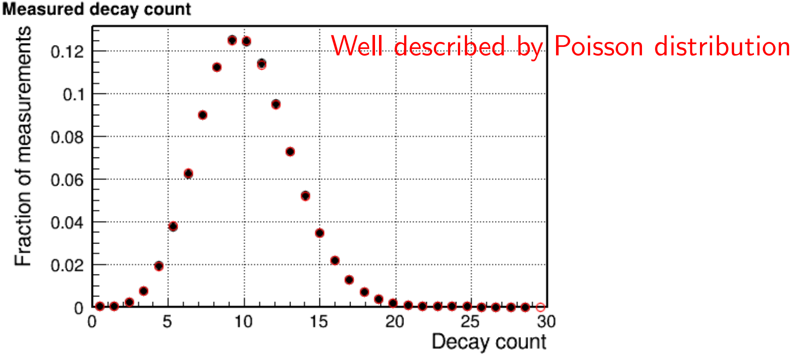

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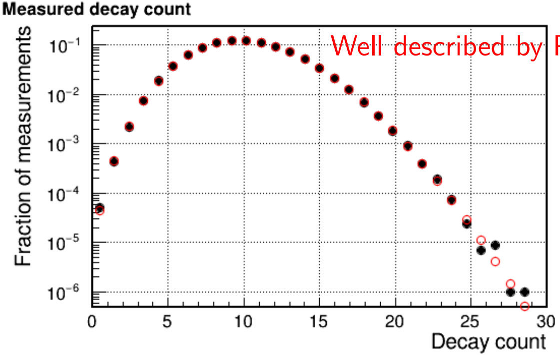


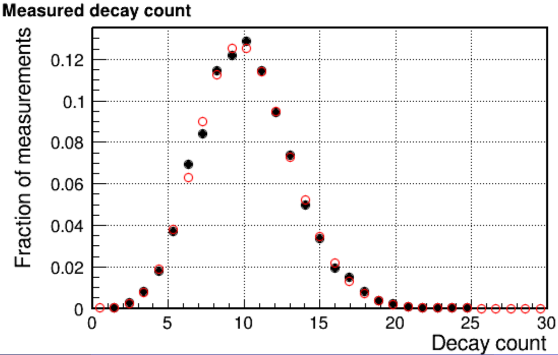
Well described by Poisson distribution

Poisson probability distribution

Formula for the Poisson probability distribution: red circles in the plot

$$P(n) = \frac{\mu^n \, e^{-\mu}}{n!} \quad \text{for } n = 0, 1, 2, \ldots$$

where μ, the expected number of events (mean), is the only parameter (!)



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Variance of the Poisson distribution

$$\mathbb{V}(n) \;=\; \langle\, (n-\mu)^2 \,\rangle \;=\; \sum_n (n-\mu)^2 \, P(n) \;=\; \mu$$

Often defines statistical uncertainty of the measurement...

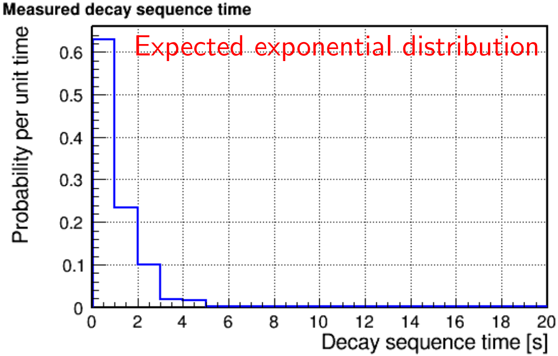
Expected time of decay sequence

03_gamma.ipynb Open in Colab

Consider radioactive source with 1 decay per second (1 Bq).

What is the expected result of the experiment measuring time needed for N decays?

Example of sequence measurements: 1 decay, 1000 measurements



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03_gamma.ipynb Open in Colab

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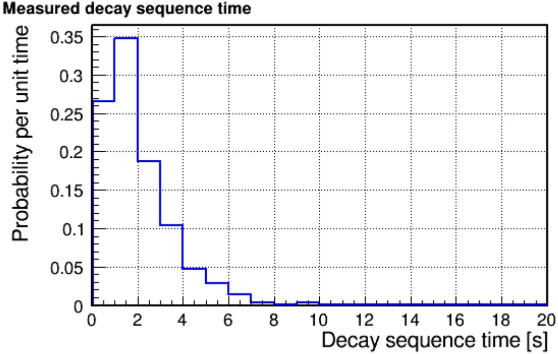
What is the expected result of the experiment measuring time needed for N decays?

Example of sequence measurements: 2 decays, 1000 measurements



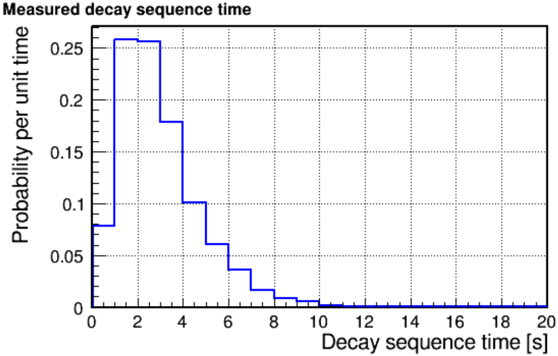
Expected time of decay sequence

03_gamma.ipynb Open in Colab

Consider radioactive source with 1 decay per second (1 Bq).

What is the expected result of the experiment measuring time needed for N decays?

Example of sequence measurements: 3 decays, 1000 measurements



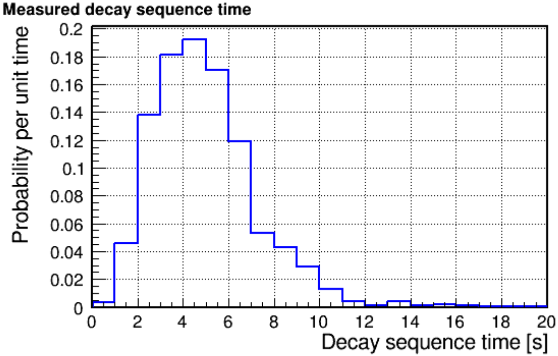
Expected time of decay sequence

03_gamma.ipynb Open in Colab

Consider radioactive source with 1 decay per second (1 Bq).

What is the expected result of the experiment measuring time needed for N decays?

Example of sequence measurements: 5 decays, 1000 measurements



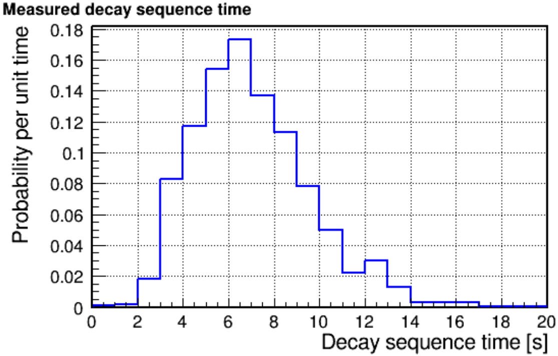
Expected time of decay sequence

03_gamma.ipynb Open in Colab

Consider radioactive source with 1 decay per second (1 Bq).

What is the expected result of the experiment measuring time needed for N decays?

Example of sequence measurements: 7 decays, 1000 measurements



Expected time of decay sequence

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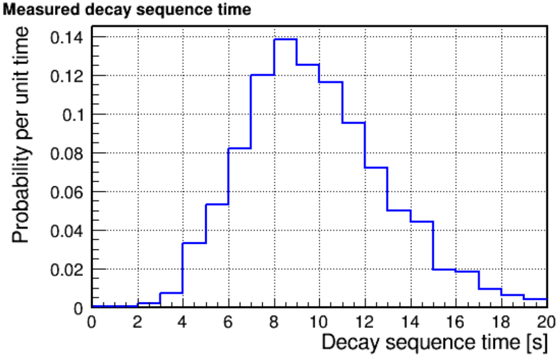
Consider radioactive source with 1 decay per second (1 Bq).

What is the expected result of the experiment measuring time needed for N decays?

Example of sequence measurements: 10 decays, 1000 measurements



Expected time of decay sequence

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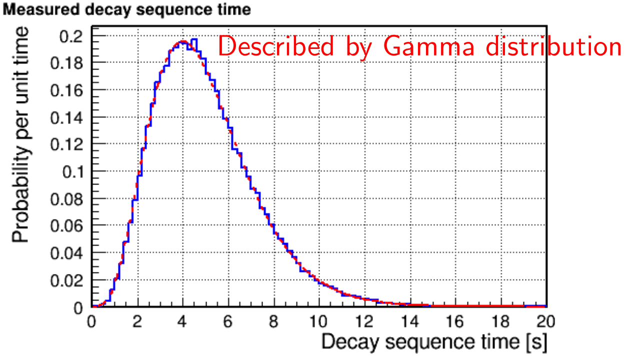
Consider radioactive source with 1 decay per second (1 Bq).

What is the expected result of the experiment measuring time needed for N decays?

Example of sequence measurements: 5 decays, 100000 measurements



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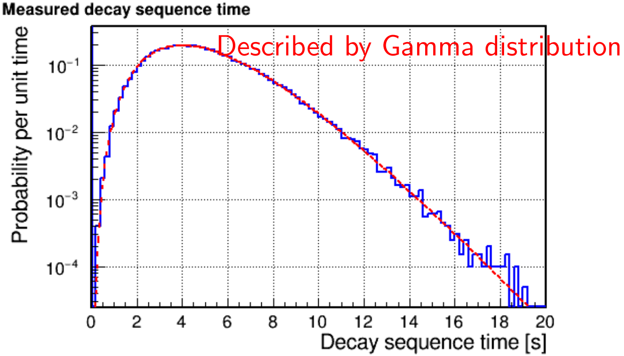

Gamma distribution

Decay sequence time distribution is described by Gamma distribution:

$$f(x) = \begin{cases} \frac{x^{k-1}\,\lambda^k\,e^{-\lambda x}}{\Gamma(k)} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

where $k \geq 0$ and $\lambda \geq 0$ are real parameters of the Gamma distribution.

For decay sequence of n decays: $k = n$ and $\lambda = 1/\tau$

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Variance of the Gamma distribution

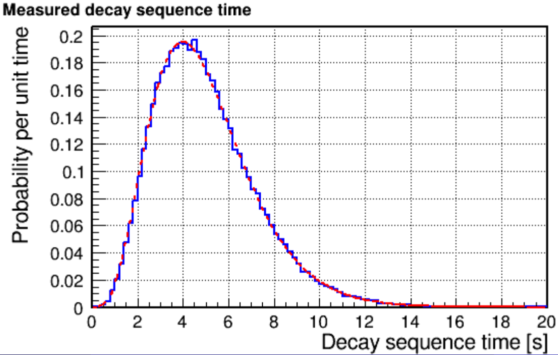
$$\mathbb{V}(x) = \sigma^2 = \frac{k}{\lambda^2}$$

Gamma distribution

For $k > 1$ $(n > 1)$ distribution has a maximum at

$$x_0 = \frac{k-1}{\lambda}$$

5 decays, 100000 measurements



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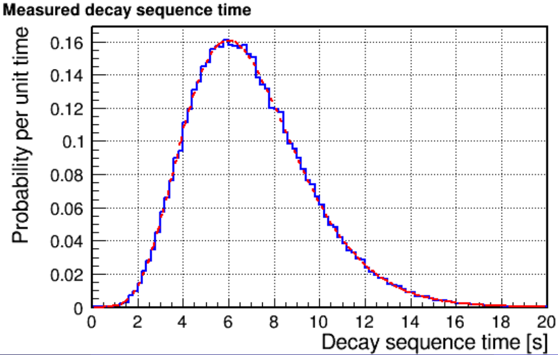

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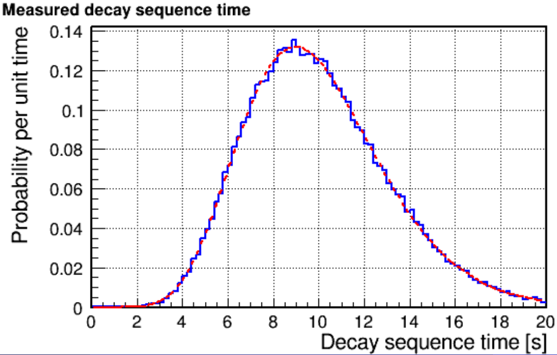

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Gamma distribution can also be written in equivalent form:

$$f(x) = A \cdot \exp\left[- \left(\frac{x_0}{\sigma_0} \right)^2 \left(\frac{x - x_0}{x_0} - \ln \frac{x}{x_0} \right) \right]$$

where A is normalization factor and σ_0 describes the width of the distribution around x_0:

$$\sigma_0^2 = \frac{k-1}{\lambda^2}$$

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Gamma distribution

Gamma distribution is a "natural" choice for describing many physical processes.
Its properties are very similar to those of the Gaussian distribution with one additional advantage: negative results are excluded by definition. One can think of the Gamma distribution as an analogue of the Gaussian one restricted to the non-negative results ($\mathbb{R}_+$)

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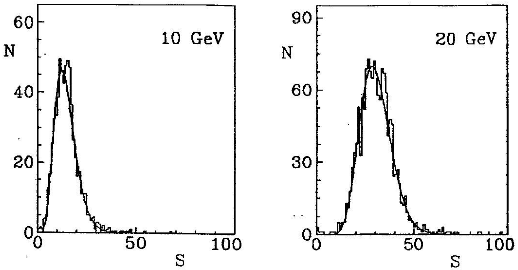
Sampling calorimeter response distribution from beam tests A.F.Żarnecki Ph.D. thesis (1992)



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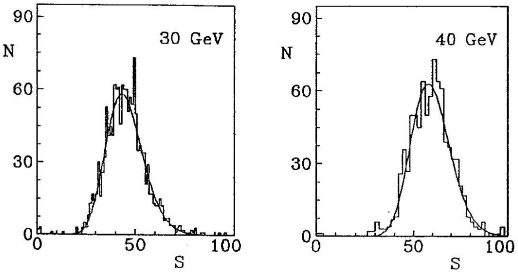

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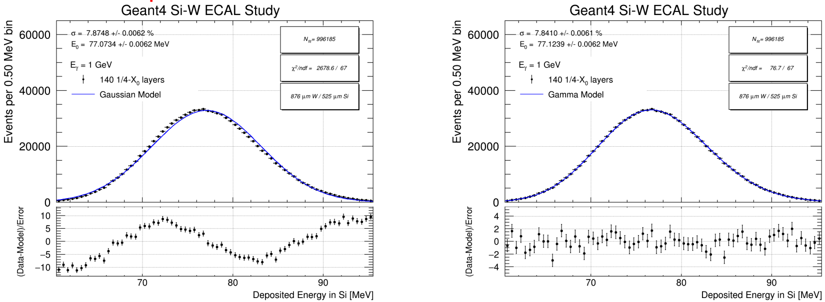
Sampling calorimeter response distribution from GEANT4 simulation Graham Wilson @ LCWS'2024



Gaussian (Normal) distribution

Is most frequently used to describe fluctuations of the measurements and resulting measurement uncertainties

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right)$$

where μ and σ are two real parameters of the distribution describing

Mean (expected value) of the Gaussian distribution

$$\mathbb{E}(x) = \langle x \rangle = \mu$$

and **Variance** of the Gaussian distribution

$$\mathbb{V}(x) = \langle (x-\mu)^2 \rangle = \sigma^2$$

Combined distributions

Consider two independent random variables X and Y:

$$f(x,y) \;=\; f_x(x)\cdot f_y(y)$$

The sum of these variables, $z = x + y$, is also a random variable and

$$\bar{z} \;=\; \mathbb{E}(Z) \;=\; \mathbb{E}(X) \;+\; \mathbb{E}(Y) \;=\; \bar{x} \;+\; \bar{y}$$

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Combined distributions

Consider two independent random variables X and Y:

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If X and Y are described by Gaussian distribution function,
then Z is also described by Gaussian probability distribution !

This is widely used when eg. describing measurement uncertainties.

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If X and Y are described by Gaussian distribution function,
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This is widely used when eg. describing measurement uncertainties.

However, this is also the case for the Gamma distribution !!!

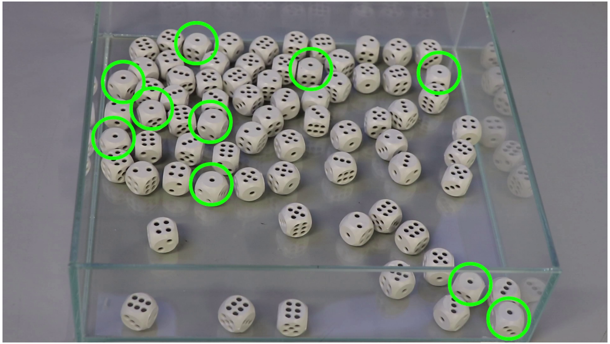
Probability distributions and their properties

Model of radioactivity

Simple way to demonstrate properties of radioactive decay...

Assume you start with a set of $N_0 = 100$ dice.

1. at each step you roll all dice you have (a step is our time unit)
2. you then remove all dice which show 1 (they "decay") and go to next step



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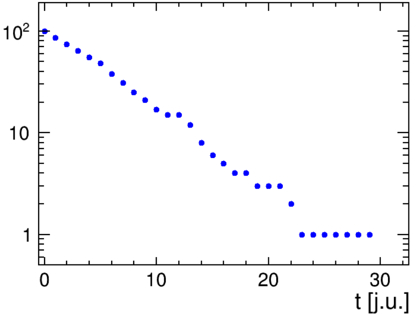
Example result:



Model of radioactivity

Can we answer the following questions:

- What is the expected (average) time for all dice to "decay"?
- What is the expected total decay time distribution?

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Step 1: calculate the effective average lifetime for a single dice.

Survival probability for $t = 1$ (one roll):

$$1 - F(t) = \exp(-\frac{t}{\tau_1}) = \frac{5}{6}$$

$$\Rightarrow \quad \tau_1 = t/\ln(1.2) \approx 5.4848$$

Model of radioactivity

Step 2: calculate the expected time for first decay in collection of N dice

Decay probability for single dice, for small time interval $dt \ll 1$:

$$dp_1 = \frac{dt}{\tau_1}$$

Model of radioactivity

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Decay probability for single dice, for small time interval $dt \ll 1$:

$$dp_1 = \frac{dt}{\tau_1}$$

For very small dt probability of having two decays within dt can be neglected.
Decay probability in collection of N dice is then given by:

$$dp_N = N \cdot dp_1 = \frac{N}{\tau_1} dt$$

$$\Rightarrow \quad \tau_N = \frac{\tau_1}{N}$$

Practical example

Model of radioactivity

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Step 3: knowing the expected waiting time for each decay,
we can calculate the expected time for all dice to decay:

$$t_{tot} = \sum_{N=N_{tot}}^{1} \frac{\tau_1}{N} = \tau_1 \cdot \sum_{N=1}^{N_{tot}} \frac{1}{N} \approx 5.1874\, \tau_1 \approx 28.45$$

Model of radioactivity

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We can also calculate variance of the total decay time by summing variances:

$$\sigma^2_{t_{tot}} = \sum_{N=N_{tot}}^{1} \left(\frac{\tau_1}{N}\right)^2 = \tau_1^2 \cdot \sum_{N=1}^{N_{tot}} \frac{1}{N^2} \approx 1.6350\ \tau_1^2 \approx 49.19$$

$$\Rightarrow \quad \sigma_{t_{tot}} \approx 7.01$$

Model of radioactivity

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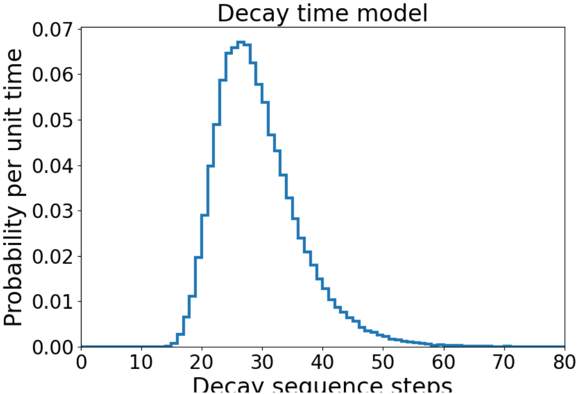
Total decay time distribution from (large number of) simulated experiment



Simulation results: 28.95 ± 6.99 in very good agreement with expectation (28.45 ± 7.01)

Practical example

Model of radioactivity

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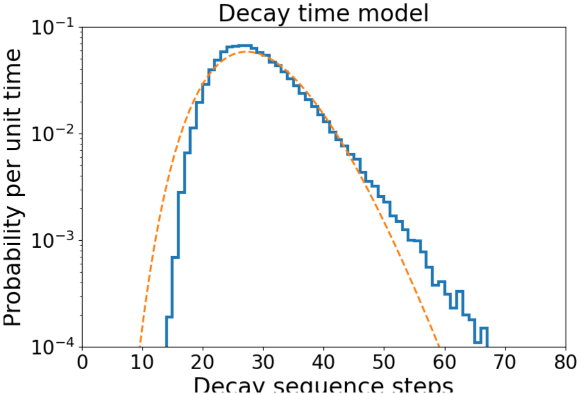

Distribution similar to the gamma distribution. Differences due to the absence of fractional decay

Probability distributions and their properties

Evaluation of branching ratios

Consider an experiment measuring decays of the τ lepton.
Events are classified into three categories:

- electron decays: $\tau^- \to e^- \nu_\tau \bar{\nu}_e$
- muon decays: $\tau^- \to \mu^- \nu_\tau \bar{\nu}_\mu$
- other decays (mostly hadronic)

After measuring 880 decays, 155 events were classified as electron and 145 as muon ones.

Calculate the resulting branching ratios (decay probabilities in given channel) and their uncertainties (!).

Solutions should be uploaded until October 30.